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Field-theoretic study of the nonlinear Fokker-Planck equation†

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Abstract. A new field-theoretic formulation of the Fokker-Planck approach to non-equilibrium statistical mechanics is presented. Starting with the nonlinear functional Fokker-Planck equation, a new generating functional is derived. No use of auxiliary conjugate fields or response functions is needed. The Feynman rules are deduced, and the renormalisation of the theory is carried out. Finally, the renormalisation group equation is solved, and scaling laws and critical exponents are calculated, which are in good agreement with previous results obtained through different formalisms.

1. Introduction

In non-equilibrium statistical mechanics the Fokker-Planck equation has been used (Garrido and San Miguel 1977, 1978, Haken 1977 Lifshitz and Pitaevskii 1981) to study the dynamics of statistical processes. Equivalently, the Langevin equation serves the same purpose: using it as a starting point, field theory techniques (de Dominicis *et al* 1975, Bausch *et al* 1976, de Dominicis and Peliti 1978) (path integral formalism, renormalisation group) have been devised to predict the behaviour of a system near its critical point. Good results have been obtained (Hohenberg and Halperin 1977), which have led us to consider the semiphenomenological Langevin approach as physically plausible. As we are going to see, the alternative Fokker-Planck equation provides us with a powerful formulation in which field theory concepts can also be used. However, in it, the stochastic force which is explicitly written in Langevin's equation is no longer present. This allows us to avoid the definition of auxiliary conjugate fields, which are commonly introduced in the most extended formalism (Martin *et al* 1973). As in that formalism, volume divergences will arise, with a specific role, when we write the generating functional. Although we are going to study a particular model (model A in Hohenberg and Halperin (1977)), that is, the time dependent Ginzburg-Landau equation, our formulation can be extended to other ones discussed in Hohenberg and Halperin (1977).

2. Hamiltonian formulation of Fokker-Planck equation

The Fokker-Planck functional equation, which we are going to analyse, can serve as a model for time evolution in ferromagnets, lasers (continuous modes), hydrodynamics,

† Based upon the PhD Thesis of A Muñoz Sudupe.

etc (Haken 1977). It has the following explicit form:

$$\frac{\partial f}{\partial t}=\int \mathrm{d}^{d}x\left(\frac{\delta}{\delta q(x)}\left(\alpha_0 q-\gamma_0\Delta q+\beta_0 q^3\right)+\frac{Q_0}{2}\frac{\delta^2}{\delta q^2(x)}\right)f \tag{1}$$

where $f(\{q(x)\}, t)$ is the probability distribution for the real order parameter $q(x)$, $Q_0 > 0$ is the diffusion coefficient and α_0, $\beta_0>0$, γ_0 characterise the drift coefficient.

As has been previously noted in dimension $d=0$ (Mühlschlegel 1978), a useful transformation involving the stationary probability distribution leads to a Hamiltonian version of (1). This transformation has not yet been extended to dimension $d\neq 0$ with the order parameter defined over a continuum. If we write $f=f_0^{1/2}\varphi$, $f_0=\exp\{-(2/Q_0)\int \mathrm{d}^d x[\frac{1}{2}\alpha_0 q^2+\frac{1}{4}\beta_0 q^4+\frac{1}{2}\gamma_0(\nabla q)^2]\}$ being the stationary probability distribution, then (1) can be cast in the following Hamiltonian form:

$$\partial\varphi/\partial t=-\int \mathrm{d}^d x\mathscr{H}\varphi \tag{2}$$

with ($\delta_0^{(d)}\equiv\delta^{(d)}(x=0)$)

$$\mathscr{H}=-\tfrac{1}{2}Q_0\,\delta^2/\delta q^2+\tfrac{1}{2}Q_0^{-1}(\alpha_0 q-\gamma_0\Delta q+\beta_0 q^3)^2 \\ -\tfrac{1}{2}\alpha_0\delta^{(d)}_{(0)}+\tfrac{1}{2}\gamma_0(\Delta\delta^{(d)}(x))_{x=0}-\tfrac{3}{2}\beta_0 q^2\delta^{(d)}_{(0)}. \tag{3}$$

The transformation $f=f_0^{1/2}\varphi$ has been extensively applied by many authors (Graham 1980, Risken 1972). When applied to a space-dependent order parameter it gives rise to the volume divergences present in (3) and, depending on the space dimension, to ultraviolet divergences. In the older formalism which starts directly from (1) those were generated by a Jacobian (Graham 1973, de Dominicis and Peliti 1978, Bausch *et al* 1976).

As we mentioned above, the volume divergences which appear in (3) are not characteristic of our model (Bausch *et al* 1976, de Dominicis and Peliti 1978). Let us see what is the role played by them: one can define creation–annihilation operators a, a^* of 'thermal modes', which decrease and increase exponentially in time. Following the field theory concepts we can write the field, in the Dirac picture, as

$$q(x,t)=\frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^d k}{\Delta_0(k)}[a(k)\exp(\mathrm{i}kx-\Delta_0 t)+a^*(k)\exp(-\mathrm{i}kx+\Delta_0 t)] \tag{4}$$

where $\Delta_0(k)=\alpha_0+\gamma_0 k^2$. Obviously the conjugate momentum is $\pi=-\mathrm{i}Q_0\delta/\delta q$, with the following commutation rule: $[q(x),\ \pi(x')]=\mathrm{i}Q_0\delta^{(d)}(x-x')$. In terms of these operators the unperturbed Hamiltonian ($\beta_0=0$) has a simple form:

$$H_0=(2/Q_0)\int \mathrm{d}^d k\, a^*(k)a(k). \tag{5}$$

The effect of the volume divergences, up to this order, has been to cancel with the 'vacuum energy' of the field. In higher orders, the $\frac{3}{2}\beta_0 q^2\delta^{(d)}(x=0)$ term in (3) ensures the correct ordering of the a, a^* operators in the Hamiltonian. These volume divergences cancel exactly after the ordering has been attained. This is not the first time that creation–annihilation operators have been introduced in this or a related context (Guyer 1982).

The formalism introduced (let us call it 'canonical') can be used to study the correlation function perturbatively. Instead of doing that, we have preferred to introduce a functional formulation.

3. Functional formalism, perturbation theory

The classical Lagrangian associated to (3) may be written as ($\dot{q} = \partial q/\partial t$)

$$\mathscr{L}(q, \dot{q}) = \tfrac{1}{2}Q_0^{-1}\dot{q}^2 + \tfrac{1}{2}Q_0^{-1}(\alpha_0 q - \gamma_0 \Delta q + \beta_0 q^3)^2 - \tfrac{1}{2}\alpha_0 \delta^{(d)}_{(0)} + \tfrac{1}{2}\gamma_0 (\Delta \delta^{(d)}_{(x)})_{x=0} - \tfrac{3}{2}\beta_0 q^2 \delta^{(d)}_{(0)}. \tag{6}$$

From this we can define the partition function or generating functional as a path integral:

$$Z(J) = N \int [\mathrm{d}q] \exp\left(- \int \mathrm{d}^d x \, \mathrm{d}t [\mathscr{L}(q, \dot{q}) + Jq] \right) \tag{7}$$

where $J(x, t)$ and N are a real source and a normalisation constant, respectively. This generating functional is not the same as the one we would have obtained directly from (1) following the work pioneered by Graham (1973) and usually employed in the literature (Bausch *et al* 1976, de Dominicis and Peliti 1978); in it, there is a linear term in the time derivative of the order parameter which is absent in (7) (see e.g. equation (2.41) in Graham (1973)). This allows us to eliminate auxiliary fields and to use field theory techniques in close analogy with the procedure followed in the static theory (Brézin *et al* 1976) and eventually leads to some simplifications. It also has important consequences if one would try to study the discretised version of (7) because there would be no coupling between the $\dot{q}$ and Δq terms.

In terms of the generating functional (7) the N-point correlation functions can be obtained as derivatives with respect to the source $J(x, t)$:

$$\langle q(x_1, t_1) \ldots . . q(x_N, t_N) \rangle = [\delta^N Z(J)/(\delta J(x_1, t_1) \ldots \delta J(x_N, t_N))]_{J=0}. \tag{8}$$

Calling $Z_0(J)$ the free generating functional, which has the unperturbed Lagrangian ($\beta_0 = 0$) in the action, and denoting by $\mathscr{L}_{\mathrm{I}}$ the interaction Lagrangian which has the form

$$\mathscr{L}_{\mathrm{I}}(q) = \tfrac{1}{2}\beta_0^2 q^6/Q + (\alpha_0 q - \gamma_0 \Delta q)\beta_0 q^3/Q - \tfrac{3}{2}\beta_0 q^2 \delta^{(d)}_{(0)} \tag{9}$$

we can formally express the partition function as

$$Z(J) = \left(\exp - \int \mathrm{d}^d x \, \mathrm{d}t \, \mathscr{L}_{\mathrm{I}}(q(x, t) \to \delta/\delta J(x, t)) \right) Z_0(J). \tag{10}$$

From this equation we calculate the expansion of the correlation functions in powers of β_0.

The path integral corresponding to $Z_0(J)$ is gaussian and thus can be easily integrated (Abers and Lee 1973) with the following result:

$$Z_0(J) = N_0 \exp \tfrac{1}{2} \int J(x, t) G_0(x - x', t - t') J(x', t') \mathrm{d}^d x \, dt \, \mathrm{d}^d x' \, \mathrm{d}t'. \tag{11}$$

The free propagator in the above expression, written in momentum space, is

$$\tilde{G}_0(k, \omega) = Q_0/[\omega^2 + (\alpha_0 + \gamma_0 k^2)^2]. \tag{12}$$

The relevance of this propagator in dealing with parabolic type equations has been recently pointed out by Faris and Jona-Lasinio (1982).

4. Feynman rules

A careful calculation, up to second order, of the perturbative series in β_0 for the correlation function shows that the Feynman rules associated to it reduce to considering just one line, corresponding to (12) and the two types of vertices drawn in figure 1.

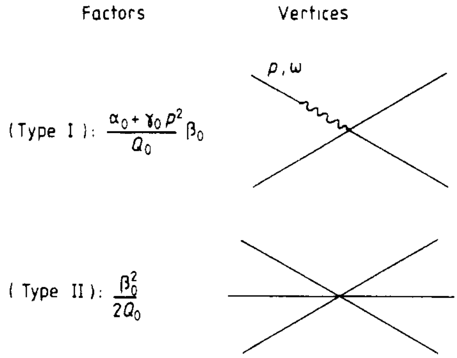

Figure 1. Types of vertices.

The effect of the volume divergences is to eliminate all the graphs with closed loops like the one drawn in figure 2. So the theory is free from volume divergences and closed loops of that kind. The expressions of the vertex functions with two and four external 'legs' are given to second order in β_0 in (13) and (19).

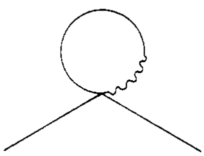

Figure 2. Closed loops cancelled by the volume divergences.

$$\begin{aligned}\Gamma^{(2)}(p,\Omega)=\tilde{G}_0^{-1}(p,\Omega)&+6(\beta_0/Q_0)\Delta_0(p)I_1-36(\beta_0^2/Q_0^2)\Delta_0(p)I_2I_1\\&+27(\beta_0^2/Q_0)I_1^2-12(\beta_0^2/Q_0)\Delta_0(p)I_4(p,\Omega)\\&-6(\beta_0^2/Q_0^2)\Delta_0^2(p)I_3(p,\Omega)-6(\beta_0^2/Q_0^2)I_5(p,\Omega)\end{aligned}\tag{13}$$

with $\Delta_0(p)=\alpha_0+\gamma_0 p^2$ and

$$I_1=\int\frac{d^dk\,d\omega}{(2\pi)^{d+1}}\tilde{G}_0(k\,\omega)\tag{14}$$

$$I_2=\int\frac{d^dk\,d\omega}{(2\pi)^{d+1}}\Delta_0(k)[\tilde{G}_0(k,\omega)]^2\tag{15}$$

$$I_3(p,\Omega)=\int\frac{d^dk\,d\omega\,d^dk'\,d\omega'}{(2\pi)^{2d+2}}\tilde{G}_0(k,\omega)\tilde{G}_0(k',\omega')\tilde{G}_0(p-k-k',\Omega-\omega-\omega')\tag{16}$$

$$I_4(p,\Omega)=\int \frac{\mathrm{d}^dk\,\mathrm{d}^dk'\,\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^{2d+2}}[\Delta_0(k)+\Delta_0(k')+\Delta_0(p-k-k')]$$
$$\times\tilde{G}_0(k,\omega)\tilde{G}_0(k',\omega')\tilde{G}_0(p-k-k',\Omega-\omega-\omega') \qquad (17)$$
$$I_5(p,\Omega)=\int \frac{\mathrm{d}^dk\,\mathrm{d}^dk'\,\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^{2d+2}}[\Delta_0(k)+\Delta_0(k')+\Delta_0(p-k-k')]^2$$
$$\times\tilde{G}_0(k,\omega)\tilde{G}_0(k',\omega')\tilde{G}_0(p-k-k',\Omega-\omega-\omega'). \qquad (18)$$

We have so far written the subscript '0' to denote bare parameters. Graphically we can represent the expression (13) as is shown in figure 3 (permutations of the lines have not been represented).

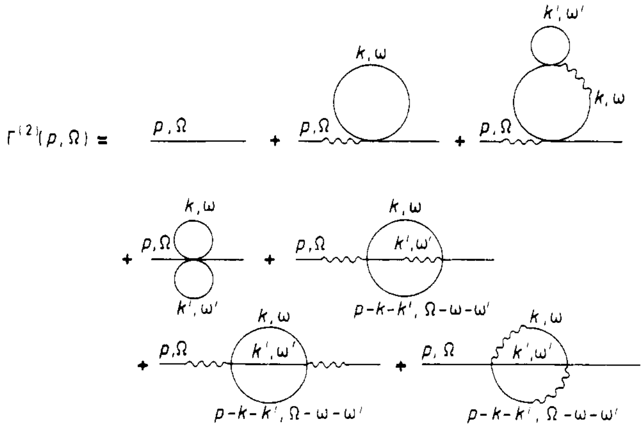

Figure 3. Two-point vertex function to order two loops.

$$\Gamma^{(4)}(p_i,\Omega_i)=6(\beta_0/Q_0)[\Delta_0(p_i)+\ldots+\Delta_0(p_4)]-36(\beta_0^2/Q_0^2)[\Delta_0(p_1)+\ldots+\Delta_0(p_4)]$$
$$\times[I_6(p_1+p_2,\Omega_1+\Omega_2)+2\mathrm{perm}]$$
$$-72(\beta_0^2/Q_0^2)[\Delta_0(p_1)+\Delta_0(p_2)][\Delta_0(p_3)+\Delta_0(p_4)]$$
$$\times[I_7(p_1+p_2,\Omega_1+\Omega_2)+2\mathrm{perm}]$$
$$-18(\beta_0^2/Q_0^2)[I_8(p_1+p_2,\Omega_1+\Omega_2)+2\mathrm{perm}]+180(\beta_0^2/Q_0)I_1 \qquad (19)$$

where

$$I_6(p,\Omega)=\int\frac{\mathrm{d}^dk\,\mathrm{d}\omega}{(2\pi)^{d+1}}\Delta_0(k)\tilde{G}_0(k,\omega)\tilde{G}_0(p+k,\Omega+\omega) \qquad (20)$$

$$I_7(p,\Omega)=\int\frac{\mathrm{d}^dk\,\mathrm{d}\omega}{(2\pi)^{d+1}}\tilde{G}_0(k,\omega)\tilde{G}_0(p+k,\Omega+\omega) \qquad (21)$$

$$I_8(p,\Omega)=\int\frac{\mathrm{d}^dk\,\mathrm{d}\omega}{(2\pi)^{d+1}}[\Delta_0(k)+\Delta_0(p+k)]^2\tilde{G}_0(k,\omega)\tilde{G}_0(p+k,\Omega+\omega). \qquad (22)$$

by 2 perm we represent in (19) two integrals in which p_1+p_2, $\Omega_1+\Omega_2$ are been replaced by p_1+p_3, $\Omega_1+\Omega_3$ and p_1+p_4, $\Omega_1+\Omega_4$ respectively. Graphically, (19) is shown in figure 4.

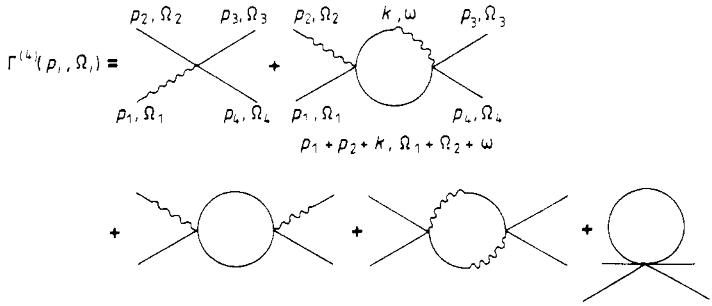

Figure 4. Four-point vertex function to order one loop.

Note that although there is a fourth power of a momentum in the denominator of (12), it does not not assure us that this theory will be less divergent than its static limit (φ_d^4) is. In fact, it is superficially more divergent. Most of the expressions given above have no meaning a sufficiently high dimension ($d \geqslant 2$). It has to be understood that all the propagators are properly regularised, for instance, we can introduce a cut-off Λ for the momentum and write the free correlation function as

$$\frac{Q_0}{\omega^2+\Delta_0^2(p)} = Q_0 \int_{(\gamma_0\Lambda^2)^{-2}} \mathrm{d}x \, \exp -x[\omega^2+\Delta_0^2(p)]. \tag{23}$$

5. Divergence degree

The divergence degrees of the 'worst' integrals ((18) and (22)) in $\Gamma^{(2)}$ and $\Gamma^{(4)}$ are $\delta=4$ and $\delta=2$, that is, quartic and quadratic repectively, in $d=4$. Clearly, these graphs are more divergent than the corresponding ones in φ_4^4 (Amit 1978, Brézin *et al* 1976) (where they are $\delta=2$ and $\delta=0$). This fact is a little puzzling because the fluctuation–dissipation theorem (Ma 1976) relates the static and dynamic vertex functions through an integral, that is

$$\int \frac{\mathrm{d}\Omega}{2\pi}[\Gamma^{(2)}(p,\Omega)]^{-1} = [\Gamma_{\mathrm{st}}^{(2)}(p)]^{-1} \tag{24}$$

and from here we are induced to believe in a similar divergence degree for both theories. As we are going to see, this is the case: the vertex of type II (see figure 1) contributes to absorbing those divergences of the dynamical theory which are larger than the corresponding ones in the static theory. At this level, both the dynamical and the static theories do have the same superficial degree of divergence.

The strategy along the renormalisation will be to follow as closely as possible the steps in the φ_d^4 theory. This can be achieved at each stage by means of the FDT (24).

The role of the FDT in renormalisation has been investigated by Deker and Haake (1975) for a time-dependent but seemingly or not explicitly space-dependent order parameter where no divergences occur. Here we point out the importance of the FDT in renormalisation for a space-dependent order parameter where divergences really arise. As in their work it allows us to formulate the perturbation theory in terms of just one propagator ('free' two-point correlation function).

We are going to renormalise the theory in its critical dimension ($d=4$). In it, the divergence degree of the 'worst' graphs (those with all the 'wavy' legs of type I vertices coupled to internal lines) is independent of the perturbative order. This can be easily seen from the expression

$$\delta=(n-E/2+1)d-4n+E+2 \tag{25}$$

where E is the number of external 'legs' and n is the total number of vertices.

In dimension $d=4$ the two-point vertex function $\Gamma^{(2)}$ is quartically divergent in Λ and the four-point vertex function $\Gamma^{(4)}$ is quadratically divergent. $\partial\Gamma^{(2)}/\partial p^2$ and $\partial\Gamma^{(2)}/\partial\Omega^2$ are both logarithmically divergent. We will need, then, four renormalisation conditions for α_0, β_0, $q(x, t)$ (as in the static theory) and Q_0 (this is the dynamical renormalisation).

We have preferred to follow the presentation of the renormalisation procedure as is done in the static case by Brézin *et al* (1976).

6. Mass renormalisation (one loop)

The expression (13), to first order in β_0, shows a quadratic divergence which comes from I_1 (14); as this is of the same kind as the corresponding static integral, we follow Brézin *et al* (1976) and Amit (1978) in order to absorb it in a redefinition of the 'mass' α_0. Let us define a renormalised α_1 as

$$\alpha_0=\alpha_1-3\beta_0 I_1. \tag{26}$$

This is the same definition that is carried out in the static φ^4 theory, as can be easily seen by integrating I_1 (14) over the internal frequency. Note that we are not using here the usual field theory parameters: this will have an important consequence in the dynamical renormalisation. Upon introducing (26) in (13), to first order, we find

$$\Gamma^{(2)}(p, \Omega, \alpha_1)=[\tilde{G}_0(p, \Omega, \alpha_1)]^{-1} \tag{27}$$

and thus the two-point vertex function is finite to the order of one loop.

7. Coupling constant renormalisation (one loop)

The integral $I_8(p, \Omega)$ in (22) is quadratically divergent in $d=4$. On the other hand, the coupling constant in φ_4^4 is only logarithmically divergent, so we hope to find some cancellation between the integrals which are quadratic in Λ in (19) (these are precisely the last two terms in (19)). But first let us introduce in (19) the renormalised 'mass' α_1, (26): this gives us new term, which is proportional to I_1 and thus also quadratic in Λ. This term together with the last one in (19) (also proportional to I_1) absorbs the quadratic divergence in I_8. To see this, note that I_1 can be written as

$$3I_1=\tfrac{1}{3}Q_0[I_8(p_1+p_2, 0)+2\ \mathrm{perm}] \tag{28}$$

as can be shown by integrating I_1 over frequencies, shifting the momentum and using the identity

$$\frac{Q_0^2}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{\Delta(k)+\Delta(p_1+p_2+k)}{\Delta(k)\Delta(p_1+p_2+k)}=I_8(p_1+p_2, 0). \tag{29}$$

In this way, we can write for $\Gamma^{(4)}$, after first-order 'mass' renormalisation,

$$\begin{aligned}\Gamma^{(4)}(p_i,\Omega_i)=&6(\beta_0/Q_0)[\Delta_1(p_1)+\ldots+\Delta_1(p_4)]-36(\beta_0^2/Q_0^2)[\Delta_1(p_1)+\ldots+\Delta_1(p_4)]\\ &\times[I_6(p_1+p_2,\Omega_1+\Omega_2)+2\text{ perm}]-72(\beta_0^2/Q_0^2)[\Delta_1(p_1)+\Delta_1(p_2)]\\ &\times[\Delta_1(p_3)+\Delta_1(p_4)][I_7(p_1+p_2,\Omega_1+\Omega_2)+2\text{ perm}]\\ &-18(\beta_0^2/Q_0^2)[I_8(p_1+p_2,\Omega_1+\Omega_2)-I_8(p_1+p_2,0)+2\text{ perm}] \qquad (30)\end{aligned}$$

with $\Delta_1(p)=\alpha_1+\gamma_0 p^2$.

The difference of integrals in the last term is finite in $d=4$. Each of them is quadratic so, as usual, the difference is naively logarithmic, but a shift in the frequency shows that it is indeed convergent.

Up to this point, expression (30) is logarithmically divergent: it is the second term where the divergence is present. This can only be absorbed by a redefinition of the bare coupling constant β_0. The static φ_4^4-theory expression, written as a dynamical one (by including a frequency integral), which renormalises the coupling constant, is

$$\beta_0=\beta_1+18\beta_1^2Q_0I_6(0,0). \qquad (31)$$

The substitution of this formula in (30) gives a $I_6(0,0)$ term which will subtract the logarithmic divergence of $I_6(p,\Omega)$. That is to say, $\Gamma^{(4)}(p_i,\Omega_i;\alpha_1,\beta_1)$ with α_1, β_1 given by (26) and (31) respectively, is finite in $d=4$. Note that in $d=3$ it would not be necessary to introduce β_1, as I_6 would be convergent.

8. Mass renormalisation (two loops)

In $\Gamma^{(2)}(p,\Omega)$ to second-order (13) for $d=4$, there are terms with quartic (I_5, I_1^2), quadratic (I_4, I_1) and logarithmic (I_3) divergences in Λ, in contrast to φ_4^4, where the two-point vertex function to second order is, at most, quadratically divergent. Our experience tells us that there must be some cancellations between the quartically divergent terms. But, first, let us discuss the second-order 'mass' and coupling constant renormalisations in (13).

The static φ_4^4 expression which defines the renormalised mass up to second order is written in an appropriate dynamical fashion (including ω integration) as

$$\alpha_0=\alpha_1-3\beta_0I_1+18(\beta_0^2/Q_0)I_4(0,0). \qquad (32)$$

The replacement of α_0 by α_1 in (13) has several consequences: (i) it gives a quartically divergent term (I_1^2); (ii) the second (with I_1) and the third terms (with I_1I_2) disappear: (iii) the contribution of $I_4(0,0)$ from (32) absorbs the quadratic divergence in $I_4(p,\Omega)$, leaving it as logarithmic.

The renormalisation of β_0, (31) does not give any new term, because (32) has eliminated from (13) all the linear terms in β_0 and at order β_0^2 the bare coupling constant can be substituted directly by β_1 (the difference would be of order β^3).

Now we are in a position to see how the quartically divergent terms in (13), after mass renormalisation, manage to absorb these divergences among themselves. First, note the equation

$$3I_1^2=Q_0^{-1}I_5(p,0) \qquad (33)$$

as can be shown by integrating I_5 over internal frequencies. This procedure gives us in (13) a difference $I_5(p, \Omega) - I_5(p, 0)$ similar to the one we found in the last term of (30). As there, the difference is able to reduce the divergence degree by four units, leaving us with a logarithmic integral. In fact, one can prove after a lengthy but simple calculation that the following equation holds:

$$I_5(p, \Omega) - I_5(p, 0) = -\Omega^2 I_3(p, \Omega). \tag{34}$$

Before going on, let us write at this level which is the expression for $\Gamma^{(2)}$, (13), in terms of α_1, β_1. By using (34), it is

$$\Gamma^{(2)}(p, \Omega; \alpha_1, \beta_1)$$
$$= \tilde{G}_0^{-1}(p, \Omega; \alpha_1) - 12(\beta_1^2/Q_0)\Delta_1(p)[I_4(p, \Omega; \alpha_1) - I_4(0, 0, \alpha_1)]$$
$$+ 6(\beta_1^2/Q_0^2)(\Omega^2 - \Delta_1^2(p))I_3(p, \Omega; \alpha_1). \tag{35}$$

9. Field-strength renormalisation

If we were in $d = 3$, equation (35) would be the final finite expression of the two-point vertex function. However, in $d = 4$, both $I_3(p, \Omega; \alpha_1)$ and the difference $I_4(p, \Omega; \alpha_1) - I_4(0, 0; \alpha_1)$ are still logarithmically divergent. This is a situation rather similar to the one we find in φ_4^4 after coupling constant and 'mass' renormalisations. There, the problem was solved by renormalising the field multiplicatively: this implies a multiplicative renormalisation of $\Gamma^{(2)}$ that absorbs logarithmic divergences. Following closely the static φ_4^4 procedure, we define the renormalised stochastic field as

$$q_R(x, t) = Z^{-12} q(x, t) \tag{36}$$

where Z, using our parameters, can be written as a dynamical expression like

$$Z = 1 + 6(\beta_1^2/Q_0\gamma_0)[\partial I_4(p, 0; \alpha_1)/\partial p^2]_{p=0}. \tag{37}$$

As usual, the integration of $I_4(p, 0)$ over internal frequencies shows that this is exactly the static expression. The 'field strength' or 'wavefunction' renormalisation (36) allows us to write for the renormalised two-point vertex function

$$\Gamma_R^{(2)}(p, \Omega; \alpha_1, \beta_1) = Z\Gamma^{(2)}(p, \Omega; \alpha_1, \beta_1). \tag{38}$$

Using (37) in (38) we find for the renormalised $\Gamma_R^{(2)}$ the following form:

$$\Gamma_R^{(2)}(p, \Omega; \alpha_1, \beta_1)$$
$$= \tilde{G}_0^{-1}(p, \Omega; \alpha_1) - 12(\beta_1^2/Q_0)\Delta_1(p)$$
$$\times \left\{ I_4(p, \Omega; \alpha_1) - I_4(0, 0; \alpha_1) - \Delta_1(p)\frac{1}{\gamma_0}\left[\frac{\partial}{\partial p^2} I_4(p, 0; \alpha_1)\right]_{p=0} \right\}$$
$$+ 6\frac{\beta_1^2}{Q_0^2}[\Omega^2 - \Delta_1^2(p)]\left\{ I_3(p, \Omega; \alpha_1) + \frac{1}{\gamma_0}\left[\frac{\partial}{\partial p^2} I_4(p, 0; \alpha_1)\right]_{p=0} \right\}. \tag{39}$$

The field strength renormalisation (36) affects also the definition of the renormalised constants α_1 and β_1. Remember that in the static case α_1 was just the value of $\Gamma_{St}^{(2)}$ for $p = 0$ and β_1 was $\Gamma_{St}^{(4)}$ for $p_i = 0$, $i = 1, \ldots, 4$, so equation (38) forces us to introduce α_2 and β defined as

$$\alpha_2 = Z\alpha_1 \qquad \beta = Z^2\beta_1. \tag{40}$$

This redefinition of the parameters absorbs in (37) a divergent term, like $\alpha_1 \ln \Lambda$, leaving as $p^2 \ldots$ the $\Delta_1(p) \ldots$ factor in the second term of (39) (see (41)). The redefinition of β_1 does not affect $\Gamma_R^{(2)}$ to order one loop, that is to order β^2, because, from (40) and (37), $\beta = \beta_1 + O(\beta_1^3)$.

The expression we get for $\Gamma_R^{(2)}$ (39) by means of the new renormalised parameters (40), that is, after all the static φ_4^4 renormalisations have been accomplished, is the following:

$$\Gamma_R^{(2)}(p, \Omega; \alpha_2, \beta)$$

$$= \tilde{G}_0^{-1}(p, \Omega; \alpha_2) - 12 \frac{\beta^2}{Q_0} \Delta_1(p) \times \left\{ I_4(p, \Omega; \alpha_2) - I_4(0, 0; \alpha_2) - p^2 \left[\frac{\partial}{\partial p^2} I_4(p, 0; \alpha_2) \right]_{p=0} \right\}$$

$$+ 6 \frac{\beta^2}{Q_0^2} [\Omega^2 - \Delta_2^2(p)] \left\{ I_3(p, \Omega; \alpha_2) + \frac{1}{\gamma_0} \left[\frac{\partial}{\partial p^2} I_4(p, 0; \alpha_2) \right]_{p=0} \right\} \quad (41)$$

with $\Delta_2(p) = \alpha_2 + \gamma_0 p^2$.

10. Dynamical renormalisation

In (41), the last term is still logarithmically divergent and has no analogue in the static φ^4 version of the theory. Whatever the dynamical renormalisation counterterm that absorbs this divergence will be, its static 'limit' (via the FDT theorem) must vanish. It seems reasonable that it would be something proportional to the difference $\Omega^2 - \Delta_2^2(p)$. Moreover, if we note that the following integral vanishes:

$$\int \frac{d\Omega}{2\pi} \frac{\Omega^2 - \Delta_2^2(p)}{[\Omega^2 + \Delta_2^2(p)]^2} = 0 \quad (42)$$

we are readily convinced that this is the kind of counterterm required to make (41) finite. In fact, (42) suggests a vanishing static 'limit' for a counterterm proportional to $\Omega^2 - \Delta_2^2(p)$. The question that has to be formulated now is the following: what kind of redefinitions of the parameters in $\tilde{G}_0^{-1}(p, \Omega; \alpha_2)$ can bring such a counterterm? It is evident that a simple redefinition of Q_0 alone would bring just a $\Omega^2 + \Delta_2^2(p)$ term, and that this is not appropriate: it will be necessary to redefine all the parameters appearing in $\tilde{G}_0^{-1}(p, \Omega; \alpha_2)$ in order to get it. This is a consequence of the particular form of the parameters that we have used: they do not coincide with the ones commonly introduced in the literature (r and g) (de Dominicis *et al* 1975, Bausch *et al* 1976), which are related to ours by $\beta = gQ/12$, $\alpha = rQ/2$, $\gamma = Q/2$. If r, g have been introduced, it will only be necessary to redefine Q, the diffusion parameter. Instead of doing that, we renormalise in our case α_2, γ_0 and Q_0 in the following way:

$$Q_0 = Q + \delta Q, \qquad \alpha_2 = \alpha\left(1 + \frac{\delta Q}{Q}\right), \qquad \gamma_0 = \gamma\left(1 + \frac{\delta Q}{Q}\right). \quad (43)$$

Upon the replacement of these in $\tilde{G}_0^{-1}(p, \Omega; \alpha_2, \gamma_0, Q_0)$ we find

$$\tilde{G}_0^{-1}(p, \Omega; \alpha_2, \gamma_0, Q_0) = \tilde{G}_0^{-1}(p, \Omega; \alpha, \gamma, Q) - [\Omega^2 - \Delta^2(p)](\delta Q)/Q^2 \quad (44)$$

with $\Delta(p) = \alpha + \gamma p^2$.

It is easy to adjust δQ so that the divergence present in the last term in (41) will be cancelled. Let us define

$$\delta Q = 6\beta^2\{I_3(0,0;\alpha,\gamma,Q)+\gamma^{-1}[\partial I_4(p,0;\alpha,\gamma,Q)/\partial p^2]_{p=0}\}. \tag{45}$$

Introducing this in $\Gamma_R^{(2)}$ expressed in terms of the renormalised α, β, γ and Q, one gets the final form which is finite in $d=4$. This is

$$\begin{aligned}\Gamma_R^{(2)}(p,\Omega;\alpha,\beta,\gamma,Q) &\\ &= \tilde{G}_0^{-1}(p,\Omega;\alpha,\gamma,Q) - 12\frac{\beta^2}{Q^2}\Delta(p)\\ &\quad\times\left\{I_4(p,\Omega)-I_4(0,0)-p^2\left[\frac{\partial I_4(p,0)}{\partial p^2}\right]_{p=0}\right\}_{\alpha,\gamma,Q}\\ &\quad+6\frac{\beta^2}{Q^2}[\Omega^2-\Delta^2(p)]\{I_3(p,\Omega)-I_3(0,0)\}_{\alpha,\gamma,Q}.\end{aligned} \tag{46}$$

This can be checked by integrating its inverse over the external frequency Ω in order to obtain the corresponding static renormalised two-point function (recall (24)). One can show that the integration of just the one-particle irreducible diagrams, with their corresponding external 'legs' (this is: a global factor $(\Omega^2+\Delta^2(p))^{-2}$), gives exactly the corresponding renormalised diagrams of the static φ_4^4 theory.

11. Normalisation conditions and critical theory

One could have introduced from the beginning the renormalised stochastic field $q_R(x,t)$ and the counterterms in the Fokker–Planck equation, writing it as (using (36) and $\alpha_0=\alpha+\delta\alpha$, $\beta_0=\beta+\delta\beta$, etc)

$$\frac{\partial f}{\partial t}=\int d^dx\left(\frac{\delta}{\delta q_R}[(\alpha+\delta\alpha)q_R-(\gamma+\delta\gamma)\Delta q_R+(\beta+\delta\beta)Zq_R^3]+\frac{Q+\delta Q}{2}\frac{1}{Z}\frac{\delta^2}{\delta q_R^2}\right)f. \tag{47}$$

The renormalisation constants $\delta\alpha$, $\delta\beta$, $\delta\gamma$, δQ and Z can be determined by imposing that the two-point and four-point vertex functions take the following values for vanishing external frequencies and momenta:

$$\Gamma_R^{(2)}(0,0)=\alpha^2/Q \tag{48}$$

$$\Gamma_R^{(4)}(0,\ldots,0)=24\beta\alpha/Q-864(\beta^2\alpha^2/Q^2)I_7(0,0) \tag{49}$$

$$\left[\frac{\partial\Gamma_R^{(2)}(p,0)}{\partial p^2}\right]_{p=0}=2\frac{\alpha\gamma}{Q}-6\frac{\beta^2\alpha^2}{Q^2}\left[\frac{\partial I_3(p,0)}{\partial p^2}\right]_{p=0} \tag{50}$$

$$\left[\frac{\partial\Gamma_R^{(2)}(0,\Omega)}{\partial\Omega^2}\right]_{\Omega=0}=\frac{1}{Q}-6\frac{\beta^2\alpha}{Q^2}\left\{2\left[\frac{\partial I_4(0,\Omega)}{\partial\Omega^2}\right]_{\Omega=0}-\alpha\left[\frac{\partial I_3(0,\Omega)}{\partial\Omega^2}\right]_{\Omega=0}\right\}. \tag{51}$$

The values of $\delta\alpha$, $\delta\beta$, $\delta\gamma$, δQ and Z that one gets from these expressions coincide with those introduced in the preceding sections.

In the critical theory ($\alpha=0$), one cannot impose values for the vertex functions at $p=\Omega=0$, because of the infrared divergences, so one is forced to fix the values of the vertex functions at p, Ω to be different, in general, from zero. This is equivalent to fixing a scale for the external momentum and frequency. Then the normalisation conditions can be written, in the critical theory, as

$$\Gamma_{\mathrm{R}}^{(2)}(p=\Omega=0;\alpha=0,\beta,\gamma,Q)=0$$

$$\left[\frac{\partial\Gamma_{\mathrm{R}}^{(2)}(p,0;\alpha=0,\beta,\gamma,Q)}{\partial p^2}\right]_{p^2=\mu^2}=2\frac{\gamma^2\mu^2}{Q}-6\frac{\beta^2}{Q^2}\gamma^2\mu^4\left(\frac{\partial I_3}{\partial p^2}\right)_{\substack{p^2=\mu^2\\ \Omega=0}}$$

$$\Gamma_{\mathrm{R}}^{(4)}(p_i,0;\alpha=0,\beta,\gamma,Q)|_{\mathrm{SP}}=18\frac{\beta\gamma\mu^2}{Q}\left(1-\frac{27\beta\gamma\mu^2}{Q}I_7(p_1+p_2,0)\right)\bigg|_{\mathrm{SP}}$$

$$\left[\frac{\partial\Gamma_{\mathrm{R}}^{(2)}(p,\Omega;\alpha=0,\beta,\gamma,Q)}{\partial\Omega^2}\right]_{\substack{p^2=\mu^2\\ \Omega=\gamma\mu^2}}=\frac{1}{Q}-12\frac{\beta^2}{Q^2}\gamma\mu^2\left(\frac{\partial I_4}{\partial\Omega^2}\right)_{\substack{p^2=\mu^2\\ \Omega=\gamma\mu^2}} \qquad (52)$$

where μ is a momentum that fixes the scale. The symbol SP means as usual (Amit 1978) $p_ip_j=(\mu^2/4)(4\delta_{ij}-1)$, $i, j=i, 2, 3, 4$.

12. Renormalisation group (critical theory)

In this section we derive scaling laws and critical exponents using the preceding field theory approach to the renormalisation group equations. The model that we are studying here has been previously treated by other authors through different formalisms (Ma and Mazenko 1975, Yahata 1974, de Dominicis *et al* 1975), so we are not presenting new results here but we are just showing that these are similar to the previously obtained ones.

As is well known, the renormalisation group equations can be derived from the independence of the bare theory on the external momentum scale introduced in fixing the normalisation conditions (52) for a critical theory. As we have seen (38) the relation between both the bare and the renormalised vertex functions with N external 'legs' is the following:

$$\Gamma_{\mathrm{R}}^{(N)}(p,\Omega;\beta,Q,\mu)=Z^{-N/2}\Gamma^{(N)}(p,\Omega;\beta_0,Q_0,\Lambda). \qquad (53)$$

Note that α_0 is no longer present in the right-hand side. This is because it is determined as a function of β_0, Q_0 and Λ (see (32)) in order to produce a renormalised theory with zero 'mass' $\alpha=0$. In the same way, neither γ_0 nor γ is present in (53): the reason is that we have fixed γ to be $\gamma=Q/2$, in order to obtain a renormalised theory with unit coefficient in the p^2 term of the static free propagator (calculated through the FDT). This fixes the γ_0 as a function of β_0, Q_0 and Λ.

Obviously the derivative of $\Gamma^{(2)}$ with respect to $\ln\mu$ vanishes, so, using (53), we can write the renormalisation group equation as

$$(\mu\,\partial/\partial\mu+W_\beta\partial/\partial\beta+\xi_QQ\partial/\partial Q-\tfrac{1}{2}N\eta_Z)\Gamma_{\mathrm{R}}^{(N)}(p,\Omega;\beta,Q,\mu)=0 \qquad (54)$$

where we have introduced, as usual, the Wilson function:

$$W_\beta=\mu(\partial\beta/\partial\mu)_{\beta_0,Q_0,\Lambda} \qquad (55)$$

and the exponent functions

$$\xi_Q=\mu(\partial\ln Q/\partial\mu)_{\beta_0Q_0\Lambda} \qquad \eta_Z=\mu(\partial\ln Z/\partial\mu)_{\beta_0,Q_0,\Lambda}. \qquad (56a,b)$$

We will check that the expressions for W_β (55) and η_z (56*b*) are the same, as was to be expected, as their corresponding static expressions. However, ξ_Q (56*a*) will define a dynamical critical exponent associated to critical relaxation times. For the sake of brevity we will only include in an appendix the calculation of ξ_Q.

The results, to the first relevant order in $d=4$, are

$$W_\beta = (q/4\pi^2)(\beta^2/Q) \tag{57}$$

$$\xi_Q = 24(16\pi^2)^{-2}(\beta^2/Q^2)[6\ln(\tfrac{4}{3})-1] \tag{58}$$

$$\eta_z = 24(16\pi^2)^{-2}\beta^2/Q^2. \tag{59}$$

We are now going to write the solution of (54) and derive the dynamical scaling laws. By dimensional considerations (Bausch *et al* 1976) we can write,

$$\Gamma_{\mathrm{R}}^{(N)}(p, \Omega; \beta, Q, \mu) = Q\mu^{d_\Gamma}(p/\mu, \Omega/Q\mu^2, \beta) \tag{60}$$

where $d_\Gamma = N+6-2N$ and $F(\cdot)$ is a dimensionless function. From (60) and the renormalisation group equation (54) it can be shown that $F(p/\mu,\ \Omega/Q\mu^2,\ \beta)$ takes the form

$$F(p/\mu, \Omega/Q\mu^2, \beta) = F(p/\bar{\mu}, \Omega/\bar{Q}\bar{\mu}^2, \bar{\beta})\exp\int_1^\rho \frac{\mathrm{d}\rho'}{\rho'}\left(d_\Gamma + \xi_Q - \frac{N}{2}\eta_z\right) \tag{61}$$

where $\bar{\mu}$, $\bar{\beta}$ and $\bar{Q}$ satisfy the following equations:

$$\rho\frac{\mathrm{d}\bar{\mu}}{\mathrm{d}\rho} = \bar{\mu} \qquad \rho\frac{\mathrm{d}\bar{\beta}}{\mathrm{d}\rho} = W_\beta(\bar{\beta}, \bar{Q}) \qquad \rho\frac{\mathrm{d}\bar{Q}}{\mathrm{d}\rho} = \xi_Q(\bar{\beta}, \bar{Q})\bar{Q} \tag{62}$$

with the initial conditions $\bar{\mu}(1) = \mu$, $\bar{\beta}(1) = \beta$, $\bar{Q}(1) = Q$. Taking into account (61), the solution $\Gamma_{\mathrm{R}}^{(N)}$ of the renormalisation group equation (54) can be written as

$$\Gamma_{\mathrm{R}}^{(N)}(p, \Omega; \beta, Q, \mu) = \rho^{d_\Gamma}\Gamma_{\mathrm{R}}^{(N)}\left(\frac{p}{\rho}\ \frac{\Omega}{(\bar{Q}/Q)\rho^2}; \bar{\beta}, Q, \mu\right)$$
$$\times\exp\int_1^\rho \frac{\mathrm{d}\rho'}{\rho'}\left(\xi_Q(\bar{\beta}, \bar{Q}) - \frac{N}{2}\eta_z(\bar{\beta}, \bar{Q})\right) \tag{63}$$

which establishes the scaling form of the N-point vertex function under a change in the external momentum scale $p \to p/\rho$.

If we denote by β^* the fixed point of W_β, that is $W_\beta(\beta^*) = 0$, it is evident from (62) that when $\rho \to 0$, $\bar{\beta}(\rho) \to \beta^*$. In this limit the scaling form (63) becomes simpler, the integrals can be explicitly done, and we obtain

$$\Gamma_{\mathrm{R}}^{(N)}(p, \Omega; \beta, Q, \mu) \underset{\rho\to 0}{\simeq} \rho^{d_\Gamma+\xi^*-N\eta^*/2}\Gamma_{\mathrm{R}}^{(N)}\left(\frac{p}{\rho}, \frac{\Omega}{\rho^{2+\xi^*}}, \beta^*, Q, \mu\right) \tag{64}$$

where we have defined $\xi^* = \xi_Q(\beta^*)$ and $\eta^* = \eta_z(\beta^*)$. Thus, in the critical region ($\rho \to 0$) equation (64) is the scaling form of the N-point vertex function; in particular, the two-point correlation function scales in the critical region as

$$\tilde{G}_{\mathrm{R}}(p, \Omega) = [\Gamma_{\mathrm{R}}^{(2)}(p, \Omega)]^{-1}$$

$$\tilde{G}_{\mathrm{R}}(p, \Omega; \beta, Q, \mu) \underset{\rho\to 0}{\simeq} \rho^{2+z-\eta^*}\tilde{G}_{\mathrm{R}}(p\rho, \Omega\rho^z, \beta^*, Q, \mu) \tag{65}$$

where we have introduced the dynamical critical exponent z defined as usual (Ma 1976) as $z = 2+\xi^*$.

3. ε expansion

We have so far worked in the unphysical dimension $d=4$, where the fixed points of the theory are trivial ($\beta^*=0$). It is useful, in order to obtain the critical exponents in a physical dimension $d=3$, to consider the expansion of the critica exponents in powers of $\varepsilon=4-d$ and β. To first order this can be easily done by introducing the dimensionless coupling constants v and v_0 defined as

$$\beta=\mu^{\varepsilon}v \tag{66}$$

$$\beta_0=\Lambda^{\varepsilon}v_0. \tag{67}$$

From the normalisation condition (52) for $\Gamma_{\mathrm{R}}^{(4)}$ and the regularised propagators (23) we find to order v_0^2 the relation

$$\mu^{\varepsilon}v=\Lambda^{\varepsilon}v_0-\frac{9}{2}\Lambda^{2\varepsilon}\frac{v_0^2Q_0}{\gamma_0^2}\int_{\Lambda^{-2}}\mathrm{d}x_1\,\mathrm{d}x_2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\exp[-x_1k^2-x_2(p_1+p_2+k)^2]|_{\mathrm{SP}}. \tag{68}$$

From here, upon integrating as in the appendix, we obtain in first order

$$W_v=-\varepsilon v+(9/4\pi^2)v^2/Q. \tag{69}$$

The fixed point v^* can be easily calculated from (69). The result is

$$v^*=(4\pi^2Q/9)\varepsilon. \tag{70}$$

This gives for the critical exponents ξ^* and η^* the well known expressions

$$\xi^*=\tfrac{1}{54}\varepsilon^2[6\ln(\tfrac{4}{3})-1] \tag{71}$$

$$\eta^*=\tfrac{1}{54}\varepsilon^2. \tag{72}$$

Both expressions coincide with previous static (η^*) and dynamic (ξ^*) calculations.

14. Conclusions and outlook

We have so far presented a new functional formulation for the stochastic approach to non-equilibrium statistical mechanics which uses as starting point the Fokker–Planck equation, and its stationary solution. The most extended formalisms up to now introduce auxiliary fields and deal both with response and correlation functions, which are inessential in our formulation. This fact has several consequences: (i) the Feynman rules are simpler in our formulation; (ii) the static renormalisation procedure can be followed very closely. Moreover, the new generating functional (5) that we introduce has a purely quadratic dependence on the time derivative of the stochastic field, while the ones used by other authors mixed both a quadratic and a linear dependence. This fact also has important consequences: it is possible to write a discretised version of the generating functional (5) which may be useful for computational purposes and, moreover, enables us to obtain correlation inequalities of the same kind as the ones found in the discretised φ^4 theory (Ising model). In particular, the kinetic part of the Lagrangian leads to a ferromagnetic coupling between nearest neighbours in the time lattice.

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Appendix

We present here the evaluation of the exponent function ξ_Q defined in (56*a*). Writing an expression analogous to (45) with $p \neq 0$, $\Omega \neq 0$ (as we are in the critical theory), the calculation of Q as a function of Q_0, β_0, Λ and μ reduces to the computation of two integrals, namely: $I_3(p^2=\mu^2, \Omega=\gamma_0\mu^2)$ and $(\partial I_4(p, 0)/\partial p^2)_{p^2=\mu^2, \Omega=0}$, properly regularised according to (23), i.e. to the evaluation of gaussian integrals. It is useful to consider the formula

$$\int \prod_{i=1}^{n} \frac{d^4K_i}{(2\pi)^4} \exp(-M_{ij}K_iK_j - 2\nu_iK_i) = \frac{1}{(2\pi)^{4n}} \left(\frac{\pi}{\det M}\right)^n \exp(M_{ij}^{-1}\nu_i\nu_j). \tag{A1}$$

Instead of calculating $I_3(p, \Omega)$ or $(\partial I_4/\partial p^2)$ directly, it is easier (Amit 1978) to compute their derivatives with respect to $(\mu/\Lambda)^2$ and integrate the final result in the limit $\Lambda \to \infty$.

We find the following results $u = (\mu/\Lambda)^2$:

$$\frac{\partial I_3(p^2=\mu^2, \Omega=\gamma_0\mu^2)}{\partial u} \underset{u\to 0}{\sim} -\frac{3}{4}\frac{Q_0^3}{\gamma_0^4}\frac{1}{(16\pi^2)^2}\ln(\tfrac{4}{3})u^{-1} \tag{A2}$$

and from here, upon integrating,

$$I_3(p^2=\mu^2, \Omega=\gamma_0\mu^2) \underset{\Lambda\to\infty}{\sim} -\frac{6}{4}\frac{Q_0^3}{\gamma_0^4}\frac{1}{(16\pi^2)^2}\ln(\tfrac{4}{3})\ln\left(\frac{\mu}{\Lambda}\right). \tag{A3}$$

For $(\partial I_4/\partial p^2)$ we find in the same way

$$\frac{1}{\gamma_0}\frac{\partial I_4(p^2=\mu^2, 0)}{\partial p^2} \underset{\Lambda\to\infty}{\sim} \frac{6}{4}\frac{Q_0^3}{(16\pi^2)^2}\frac{1}{\gamma_0^4}\ln\frac{\mu}{\Lambda}. \tag{A4}$$

This is the same expression that is obtained in the static φ^4 theory for the critical exponent η_z.

Finally, adding the partial results (A3) and (A4), we get

$$Q = Q_0 + \frac{6\beta_0^2Q_0^3}{4(16\pi^2)^2}\frac{1}{\gamma_0^4}[6\ln(\tfrac{4}{3}) - 1]\ln\left(\frac{\mu}{\Lambda}\right). \tag{A5}$$

We are now ready to evaluate, ξ_Q using (56*a*). The result is

$$\xi_Q = 24(16\pi^2)^{-2}(\beta^2/Q^2)[6\ln(\tfrac{4}{3}) - 1]. \tag{A6}$$

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